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Weighted Means and Oscillation Conditions for Second Order Matrix Differential Equations

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1. INTRODUCTION

We are concerned with the second order system of differential equations

$$[P(t)\,y'(t)]' + Q(t)\,y(t) = 0 \qquad \text{for} \quad t \in [0, \infty), \tag{1.1}$$

where P and Q are $n \times n$ real, symmetric, matrix-valued functions on $[0, \infty)$ and y is an n-column vector. We also assume that $P(t)$ is positive definite for all t so that $P(t)^{-1}$ is defined. Two points $\alpha, \beta \in [0, \infty)$ are said to be conjugate to each other if there exists a real-valued, nontrivial solution $y(t)$ of (1.1) such that $y(\alpha) = y(\beta) = 0$. The system (1.1) is said to be oscillatory at infinity if for any $\alpha \in [0, \infty)$, there exists a conjugate point $\beta > \alpha$.

The oscillation theory of (1.1) has received considerable attention; see for example Reid [13, Chap. V]. It has been conjectured, see Hinton and Lewis [7], that when $P(t) \equiv I$, the identity n $\times$ n matrix, (1.1) is oscillatory at infinity if

$$\lim_{t \to \infty} \lambda_1\{Q_1(t)\} = \infty, \tag{1.2}$$

where

$$Q_1(t) := \int_0^t Q(s)\,ds \tag{1.3}$$

and $\lambda_1\{\cdot\}$ denotes the largest eigenvalue of the matrix argument. We also denote by $\lambda_n\{\cdot\}$ the least eigenvalue of a matrix.

A number of authors have tried to prove the conjecture with only partial success [1, 4, 8, 11,12]. The two main obstacles may be stated as follows.

(A) There is no restriction on the behaviour of the other eigenvalues which, far from tending to infinity, may be very large and negative.

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(B) The direction of the eigenvector of $Q_1(t)$ associated with the largest eigenvalue is not fixed and will, in general, vary considerably for different values of t.

These authors were able to give a proof of the Hinton, Lewis conjecture under mild extra conditions involving either the growth of the trace of $-Q_1(t)$ or the size of $\lambda_n\{Q_1(t)\}$. The conjecture was recently proved by Kaper and Kwong [8] for the case $n=2$. Their proof can, although not in an obvious way, be extended to establish the conjecture for general n.

The approach used in [8] was that of differential equations and involved very little matrix theory. In consequence the proof is not as short or transparent as in the scalar case. In this article, by drawing more from matrix theory, we are able not only to extend the results of [8] but also to give a proof which resembles more closely the proof of the scalar case. In particular our proof resembles in some ways that of Coles [5]. This forms a nice example of the results from one branch of mathematics being used in another.

Most of the results from matrix theory are fairly elementary, but not well known outside that field. For this reason we state these results in Section 4 and give the proofs in the Appendix.

Our approach enables us to obtain extensions of the Hinton, Lewis conjecture in two directions. First, to the case of a general, positive definite $P(t)$ in (1.1) and second, to the case in which the condition of (1.2), (1.3) is replaced by ones which are considerably more general. We establish an analogue of a theorem of Wintner [14] concerning the mean of $Q_1(t)$ and an analogue of the results of Coles and Willett [6] involving iterated Cesaro like means of $Q_1(t)$.

2. The Results

We prove the following theorems:

THEOREM 1. *Let* $S_\lambda := \{t \in [0, \infty): \lambda_1\{Q_1(t)\} \geq \lambda\}$. *If*

$$\limsup_{\lambda \to \infty} \lambda \left(\int_{S_\lambda} \lambda_n\{P(t)^{-1}\}\, dt \right) > n \tag{2.1}$$

then (1.1) *is oscillatory.*

COROLLARY 1. *If* $\int_{S_\lambda} \lambda_n\{P(t)^{-1}\}\, dt = \infty$ *for all* λ *then* (1.1) *is oscillatory.*

It is easy to see that (2.1) is implied by the conditions (1.2) and the condition

$$\int_0^\infty \lambda_n\{P(t)^{-1}\}\,dt=\infty. \tag{2.2}$$

We then not only have an analogue of the well-known Fite, Leighton, Wintner theorem, but also an analogue of the improvement first given in [14] for the scalar case.

In the next theorem we assume that $P(t)\equiv I$. A function $f\colon [0,\infty)\to\mathbb{R}$ is said to tend to infinity in the weak sense if there exists a subset S of $[0,\infty)$ such that

$$\int_S \frac{dt}{t}=\infty, \tag{2.3}$$

$$\lim_{\substack{t\in S\\ t\to\infty}} f(t)=\infty. \tag{2.4}$$

Note that $\lim_{t\to\infty} g(t)=\infty$ implies that $g(t)\to\infty$ in the weak sense.

THEOREM 2. *If $P(t)\equiv I$ and*

$$\lambda_1\left\{\frac{1}{t}\int_0^t Q_1(s)\,ds\right\}\to\infty \qquad \textit{in the weak sense}, \tag{2.5}$$

then (1.1) *is oscillatory.*

Our next result provides a partial extension of Theorem 2 to more general means and gives a result comparable to that of [6] relating to Cesaro-like means.

Let $f_1,\ldots,f_N$ be nonnegative, locally integrable, scalar-valued functions defined on $[0,\infty)$. If H denotes an $n\times n$ matrix-valued function we define the following functions:

$$\begin{aligned} I_1(t;H) &:= \int_0^t f_1(s)\,H(s)\,ds,\\ I_k(t;H) &:= \int_0^t f_k(s)\,I_{k-1}(s;H)\,ds \qquad \text{for}\quad k=2,\ldots,N; \end{aligned} \tag{2.6}$$

$$\begin{aligned} j_1(t) &:= \int_0^t f_1(s)^2\,ds,\\ j_k(t) &:= \int_0^t f_k(s)^2\,f_{k-1}(s)^{-1}\,j_{k-1}(s)\,ds \qquad \text{for}\quad k=2,\ldots,N; \end{aligned} \tag{2.7}$$

$$\begin{aligned} i_1(t) &:= \int_0^t f_1(s)\, ds, \\ i_k(t) &:= \int_0^t f_k(s)\, i_{k-1}(s)\, ds \qquad \text{for} \quad k = 2, ..., N; \end{aligned} \tag{2.8}$$

$$J(t) := \int_0^t f_N(s)\, j_N(s)^{-1}\, ds; \tag{2.9}$$

$$K(t) := i_N(t) \int_t^\infty f_N(s)\, j_N(s)^{-1}\, ds; \tag{2.10}$$

We say that $f_1, ..., f_N \in \mathscr{F}$ if the functions defined by (2.7) and (2.8) exist and either

$$J(t) \to \infty \qquad \text{as} \quad t \to \infty, \tag{2.11}$$

or

$$\lim_{t \to \infty} \sup K(t) > 0. \tag{2.12}$$

It may readily be shown, see [6, Examples 1 and 2], that

$$\left\{1, \frac{1}{t}, ..., \frac{1}{t}\right\} \in \mathscr{F} \tag{2.13}$$

and

$$\{1, 1, ..., 1\} \in \mathscr{F}. \tag{2.14}$$

THEOREM 3. *If $P(t) \equiv I$ and for some positive integer N there exists $\{f_1, ..., f_N\} \in \mathscr{F}$ such that*

$$\lim_{t \to \infty} \lambda_1 \{i_N(t)^{-1}\, I_N(t; Q_1)\} = \infty$$

then (1.1) *is oscillatory.*

COROLLARY 2. *If for some positive integer N.*

$$\lim_{t \to \infty} \lambda_1 \left\{\frac{1}{t_N} \int_0^{t_N} \frac{1}{t_{N-1}} \int_0^{t_{N-1}} \cdots \frac{1}{t_2} \int_0^{t_2} \int_0^{t_1} Q(s)\, ds\, dt_1 \cdots dt_N\right\} = \infty$$

then (1.1) *is oscillatory.*

Proof. This follows from (2.13) and Theorem 3.

COROLLARY 3. If for some positive integer N

$$\lim_{t \to \infty} \lambda_1 \left\{ \frac{1}{t_N^{N-1}} \int_0^{t_N} \int_0^{t_{N-1}} \cdots \int_0^{t_1} Q(s)\, ds\, dt_1 \cdots dt_N \right\} = \infty$$

then (1.1) *is oscillatory.*

Proof. This follows from (2.14) and Theorem 3.

3. PRELIMINARY RESULTS

The proof of our results depends on some well-known facts concerning the oscillation of (1.1). Instead of looking at (1.1) directly we can consider the corresponding matrix differential equation

$$[P(t)\, y(t)']' + Q(t)\, y(t) = 0 \qquad \text{for} \quad t \in [0, \infty) \tag{3.1}$$

in which the unknown is the $n \times n$ matrix-valued function $y(t)$. The system (1.1) is oscillatory at infinity if and only if for any conjoined solution $y(t)$ of (3.1) $\det(y(t))$ has arbitrarily large zeros. The change of variable

$$R(t) := -P(t)\, y'(t)\, y(t)^{-1} \tag{3.2}$$

transforms (3.1) into the Ricatti equation

$$R'(t) = Q(t) + R(t)\, P(t)^{-1} R(t). \tag{3.3}$$

If $R(t)$ is symmetric for some value of t then it is symmetric for all t. To see this suppose that $R(t)$ is a solution of (3.3) and $R(a)$ is symmetric. By taking the transpose of both sides of (3.3) it is easy to see that $R(t)^{\mathrm{T}}$ is also a solution, where the superscript T denotes transposition. Since $R(a)$ is real symmetric $R(t)$ and $R(t)^{\mathrm{T}}$ take the same value at a. Hence by the uniqueness theorem $R(t) \equiv R(t)^{\mathrm{T}}$.

Solutions $y(t)$ of (3.1) for which $R(t)$ is symmetric are known as conjoined or prepared solutions.

It is known that the oscillation of (1.1) is equivalent to the condition that no solution of (3.3) extends to the semi-infinite interval $[0, \infty)$. In the scalar case this fact is fundamental. Its extension to a system may be found in [8].

Our plan of attack is to assume that the system (3.1) is nonoscillatory under the given hypothesis, then there must exist a solution $R(t)$ of (3.3) defined on $[a, \infty)$ (without loss of generality we may take a to be zero). By integrating (3.3) we obtain an integral equation. We estimate the growth of the quadratic term on the right hand side. This estimate forces the matrix $R(t)$ to become arbitrarily large at a finite point and yields a contradiction.

4. Matrix Theory

Unless stated otherwise capital letters will be used throughout this section to denote Hermitian $n \times n$ matrices. The superscript * denotes the complex conjugate transpose of a matrix. The notation $\operatorname{tr}[A]$ denotes the trace, $\sum_{i=1}^{n} a_{ii}$, of the matrix A.

An Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is positive semi-definite (positive definite) if for all $u \in \mathbb{C}^n$, $u \neq 0$, the $\mathbb{C}^n$-inner product $(u \cdot Au)$ is nonnegative (positive). The notation $A \geqslant 0$ $(A > 0)$ will be used to denote a positive semi-definite (positive definite) matrix. We impose a partial ordering on the set of Hermitian $n \times n$ matrices by the relation $A \geqslant B$ if $A - B \geqslant 0$. Some properties of this definition are as follows.

If $A \geqslant 0$ then all eigenvalues of A are nonnegative. (4.1)

If $A \geqslant B$ then $\operatorname{tr}[A] \geqslant \operatorname{tr}[B]$. (4.2)

If $A > B$ and $C = C^*$ then $CAC > CBC$. (4.3)

If $0 \leqslant A$ and $A \nleqslant I$ then $\operatorname{tr}[A] > 1$. (4.4)

If $A \geqslant B$ and $C \geqslant D$ then $A + C \geqslant B + D$. (4.5)

If $A \nleqslant B$ and $B \geqslant C$ then $A \nleqslant C$. (4.6)

It may be observed that the relation "$<$" defined above is only a partial ordering and so the relations "$<$" and "$\ngeqslant$" are not equivalent.

A matrix-valued function $A(\cdot) = (a_{i,j}(\cdot)\colon (a, b) \to \mathbb{C}^{n \times n})$ is said to belong to the space $L^p(a, b)$ if and only if all of the integrals $\int_a^b |a_{i,j}(t)|^p \, dt$ are finite. We define differentiability and continuity of a matrix-valued function in a similar, componentwise, fashion.

The following observations are readily verified.

If $A(t) \geqslant 0$ for $t \in (a, b)$ then $\int_a^b A(t)\, dt \geqslant 0$. (4.7)

If $A(0) \geqslant 0$ and $A'(t) \geqslant 0$ then $A(t) \geqslant 0$ (4.8)

and

$$\int_a^b \operatorname{tr}\{A(t)^{-1} A'(t) A(t)^{-1}\}\, dt = \operatorname{tr}\{A(a)^{-1}\} - \operatorname{tr}\{A(b)^{-1}\}. \qquad (4.9)$$

It is necessary to be particularly cautious in manipulating inequalities of Hermitian matrices since multiplication by positive matrices does not, in general, preserve the order. A simple example of this is the relation

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \geqslant \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geqslant 0,$$

but

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2 \not\geqslant \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^2.$$

Fortunately we have the following, a simple case of an elegent result of Loewner [10].

If A and B are positive semi-definite and $A^2 \geqslant B^2$ then $A \geqslant B$.

Simpler proofs of this result may be found in Bellman [3], Au-Yeung [2], and Kwong [9]. A very simple proof has recently been discovered and will be reproduced in the Appendix. In this paper we require the following generalisation.

LEMMA 1. *Let $C = C^* > 0$, $A = A^* \geqslant 0$, and $B = B^*$. If $ACA \geqslant BCB$ then $A \geqslant B$.*

We note in particular the negation of Lemma 1,

$$\text{If } A \geqslant 0 \text{ and } A \not\geqslant B \text{ then } ACA \not\geqslant BCB. \tag{4.10}$$

Finally we need an analogue of the Schwarz inequality.

LEMMA 2. *If $f\colon (a, b) \to \mathbb{R}$ and $A\colon (a, b) \to \mathbb{C}^{n \times n}$ are both members of $L^2(a, b)$ then*

$$\left(\int_a^b f(t)\, A(t)\, dt\right)^2 \leqslant \left(\int_a^b f(t)^2\, dt\right)\left(\int_a^b A(t)^2\, dt\right).$$

The proofs of Lemmas 1 and 2 are given in the Appendix.

5. PROOF OF THEOREM 1

We integrate Eq. (3.3) between 0 and t to obtain

$$R(t) = R(0) + Q_1(t) + \int_0^t R(s)\, P(s)^{-1}\, R(s)\, ds. \tag{5.1}$$

We define

$$F(t) := R(0) + Q_1(t), \tag{5.1}$$

$$A(t) := \int_0^t R(s)\, P(s)^{-1}\, R(s)\, ds.$$

It may be seen that

$$A(0)=0, \qquad A(t)\geqslant 0, \qquad A'(t)\geqslant 0. \tag{5.2}$$

Moreover, for $t\in S_\lambda$, $\lambda_1\{Q_1(t)\}\geqslant\lambda$ so, for all $\delta\in(0,1)$ we may arrange, by taking λ sufficiently large, that

$$F(t)\nleqslant\delta\lambda I \qquad \text{for} \quad t\in S_\lambda.$$

It follows that

$$F(t)+A(t)\nleqslant\delta\lambda I+A(t) \qquad \text{for} \quad t\in S_\lambda \tag{5.3}$$

whence, by (4.10)

$$\begin{aligned} A'(t) &= (F(t)+A(t))\,P(t)^{-1}\,(F(t)+A(t)) \\ &\nleqslant (\delta\lambda I+A(t))\,P(t)^{-1}\,(\delta\lambda I+A(t)) \end{aligned}$$

and

$$(\delta\lambda I+A(t))^{-1}\,A'(t)(\delta\lambda I+A(t))^{-1}\nleqslant P(t)^{-1}. \tag{5.4}$$

It follows from (5.4) that for $t\in S_\lambda$

$$\operatorname{tr}\{(\delta\lambda I+A(t))^{-1}\,A'(t)(\delta\lambda I+A(t))^{-1}\}>\lambda_n\{P(t)^{-1}\}. \tag{5.5}$$

From (4.9), (5.2), and (5.5) we have that

$$\begin{aligned} \operatorname{tr}\{(\delta\lambda I+A(0))^{-1}\} &> \int_0^\infty \operatorname{tr}\{(\delta\lambda I+A(t))^{-1}\,A'(t)(\delta\lambda I+A(t))^{-1}\}\,dt \\ &\geqslant \int_{S_\lambda} \operatorname{tr}\{(\delta\lambda I+A(t))^{-1}\,A'(t)(\delta\lambda I+A(t))^{-1}\}\,dt \\ &\geqslant \int_{S_\lambda} \lambda_n\{P(t)^{-1}\}\,dt. \end{aligned} \tag{5.6}$$

By (5.2) and (5.6)

$$n>\delta\lambda\int_{S_\lambda}\lambda_n\{P(s)^{-1}\}\,ds. \tag{5.7}$$

We let $\lambda\to\infty$ in (5.7) and obtain a contradiction.

6. Proof of Theorem 2

Although the proof of Theorem 2 may be contained in an expanded version of the proof of Theorem 3, we present it separately without all the technical details necessitated by the generality of the hypotheses of Theorem 3.

Let S be a set such that

$$\lambda_1\left\{\frac{1}{t}\int_0^t Q_1(s)\,ds\right\}\to\infty \qquad \text{as} \quad t\in S\to\infty.$$

and define

$$F(t):=R(0)+\frac{1}{t}\int_0^t Q_1(s)\,ds,$$

$$A(t):=\int_0^t \frac{1}{s}\left(\int_0^s R(\xi)\,d\xi\right)^2 ds.$$

Integration of (5.1) in the case $P\equiv I$ gives

$$\begin{aligned}\frac{1}{t}\int_0^t R(s)\,ds &= F(t)+\frac{1}{t}\int_0^t\int_0^s R(\xi)^2\,d\xi\,ds\\ &\geqslant F(t)+\frac{1}{t}A(t)\end{aligned}\tag{6.1}$$

by Lemma 2. For $t\in S$ sufficiently large, we have from (6.1) that

$$\frac{1}{t}\int_0^t R(s)\,ds \not\leqslant \frac{1}{t}\,[I+A(t)].\tag{6.2}$$

It follows from (4.10) and (6.2) that for $t\in S$ sufficiently large

$$A'(t)\not\leqslant\frac{1}{t}\,[I+A(t)]^2$$

whence,

$$\{I+A(t)\}^{-1}A'(t)\{I+A(t)\}^{-1}\not\leqslant\frac{1}{t}\,I$$

and, from (4.4),

$$\operatorname{tr}\{(I+A(t))^{-1}A'(t)(I+A(t))^{-1}\}>\frac{1}{t}.\tag{6.3}$$

As in the proof of Theorem 1 we have

$$\begin{aligned}\operatorname{tr}\{(I+A(0))^{-1}\} &> \int_0^\infty \operatorname{tr}\{(I+A(t))^{-1} A'(t)(I+A(t))^{-1}\}\,dt \\ &\geqslant \int_S \operatorname{tr}\{(I+A(t))^{-1} A'(t)(I+A(t))^{-1}\}\,dt \\ &> \int_S \frac{dt}{t}\end{aligned}$$

The contradiction now follows.

7. Proof of Theorem 3

We suppose that the f_j are continuous for $j=1,\ldots,N$. The modifications for the integrable case are trivial.

Successive multiplications by the f_j and integrations transform equation (5.1) into

$$I_N(t;R)=i_N(t)\,R(0)+I_N(t;Q_1)+I_N\left(t;\int_0^t R(s)^2\,ds\right). \tag{7.1}$$

It follows from (7.1) and the hypothesis

$$\lim_{t\to\infty} \lambda_1\{i_N(t)^{-1}\,I_N(t;Q_1)\}=\infty$$

that for any $\mu>0$ we may arrange, by taking t sufficiently large, that

$$I_N(t;R) \not\leqslant i_N(t)\,\mu I+I_N\left(t;\int_0^t R(s)^2\,ds\right)$$

whence, by (4.10),

$$I_N(t;R)^2 \not\leqslant \left\{i_N(t)\,\mu I+I_N\left(t;\int_0^t R(s)^2\,ds\right)\right\}^2. \tag{7.2}$$

We now bound the quadratic term on the right-hand side of (7.2). For any $n\times n$ matrix valued-function, H, it follows from Lemma 2 that

$$I_1(t;H)^2=\left(\int_0^t f_1(s)\,H(s)\,ds\right)^2 \leqslant \left(\int_0^t f_1(s)^2\,ds\right)\left(\int_0^t H(s)^2\,ds\right). \tag{7.3}$$

Also, for $1 < m \leqslant N$,

$$\begin{aligned} I_m(t; H)^2 &= \left(\int_0^t f_m(s)\, I_{m-1}(s)\, ds\right)^2 \\ &= \left(\int_0^t f_m(s)\, \frac{j_{m-1}(s)^{1/2}}{f_{m-1}(s)^{1/2}} \cdot \frac{f^{m-1}(s)^{1/2}}{j^{m-1}(s)^{1/2}}\, I_{m-1}(s; H)\, ds\right)^2 \\ &\leqslant \left(\int_0^t f_m(s)^2\, \frac{j_{m-1}(s)}{f_{m-1}(s)}\, ds\right)\left(\int_0^t \frac{f_{m-1}(s)}{j_{m-1}(s)}\, I_{m-1}(s; H)^2\, ds\right) \\ &= j_m(t) \int_0^t \frac{f_{m-1}(s)}{j_{m-1}(s)}\, I_{m-1}(s; H)^2\, ds. \end{aligned} \tag{7.4}$$

Inductively from (7.3) and (7.4) we see that

$$\begin{aligned} j_m(t)^{-1}\, I_m(t; H)^2 \leqslant \int_0^t f_{m-1}(t_{m-1}) \int_0^{t_{m-1}} \cdots \int_0^{t_2} f_1(t_1) \\ \times \int_0^{t_1} H(s)^2\, ds\, dt_1 \cdots dt_{m-1}. \end{aligned} \tag{7.5}$$

We may rewrite (7.5) as

$$f_m(t)\, j_m(t)^{-1}\, I_m(t; H)^2 \leqslant I'_m\left(t; \int_0^t H(s)^2\, ds\right). \tag{7.6}$$

Let $A(t) := i_N(t)\, \mu I + I_N(t; \int_0^t R(s)^2\, ds)$. The relation (7.2) implies that

$$f_N(t)\, j_N(t)^{-1}\, I_N(t; R)^2 \not\leqslant f_N(t)\, j_{N-1}(t)^{-1}\, A(t)^2$$

from which by (7.6), with $H := R$ and $m := N$, and (4.6) we deduce that

$$\begin{aligned} f_N(t)\, j_N(t)^{-1}\, A(t)^2 &\not\geqslant I'_N\left(t; \int_0^t R(s)^2\, ds\right) \\ &\not\geqslant I'_N\left(t; \int_0^t R(s)^2\, ds\right) + \mu i'_N(t)\, I \\ &= A'(t). \end{aligned} \tag{7.7}$$

It follows from (7.7) that

$$A(t)^{-1}\, A'(t)\, A(t)^{-1} \not\leqslant f_N(t)\, j_N(t)^{-1}\, I$$

whence,

$$\operatorname{tr}\{A(t)^{-1}\, A'(t)\, A(t)^{-1}\} > f_N(t)\, j_N(t)^{-1}. \tag{7.8}$$

We integrate both sides of (7.8) over $[0, t)$ and obtain, by (4.9)

$$\operatorname{tr}\{A(0)^{-1}\} > \int_0^t f_N(s)\, j_N(s)^{-1}\, ds. \tag{7.9}$$

In the case that $f_1,\dots, f_N$ satisfy (2.11), (7.9) gives a contradiction. Alternatively we may integrate (7.8) over (t, ∞) to obtain

$$\operatorname{tr}\{A(t)^{-1}\} > \int_t^\infty f_N(s)\, j_N(s)^{-1}\, ds.$$

so that

$$1 > [\operatorname{tr}(A(t)^{-1})]^{-1} \int_t^\infty f_N(a)\, j_N(s)^{-1}\, ds. \tag{7.10}$$

Now,

$$A(t) = \mu i_N(t)\, I + I_N\left(t; \int_0^t R(s)^2\, ds\right)$$

so

$$A(t) \geqslant \mu i_N(t)\, I$$

and

$$A(t)^{-1} \leqslant (\mu i_N(t))^{-1}\, I.$$

Thus, by (4.2)

$$\operatorname{tr}\{A(t)^{-1}\} \leqslant n\mu^{-1} i_N(t)^{-1}$$

and

$$[\operatorname{tr}\{A(t)^{-1}\}]^{-1} \geqslant \frac{1}{n}\, \mu i_N(t). \tag{7.11}$$

From (7.10) and (7.11)

$$1 > n^{-1}\mu i_N(t) \int_t^\infty f_N(s)\, j_N(s)^{-1}\, ds$$

and

$$\frac{n}{\mu} > i_N(t) \int_t^\infty f_N(s)\, j_N(s)^{-1}\, ds. \tag{7.12}$$

Since $\mu > 0$ is arbitrary (7.12) contradicts (2.12) and the proof is complete.

8. Appendix

We collect here, for the convenience of reference, the proofs of Lemmas 1 and 2.

Proof of Lemma 1. First, we give a simple proof of the particular case of Loewner's result which we require. Suppose $A^2 \geqslant B^2$ and $A \geqslant 0$. We need only prove the case $A > 0$ since the general case follows by a continuity argument. Let $X = A - B$ then

$$0 \leqslant A^2 - B^2 = A^2 - (A - X)^2 = AX + XA - X^2.$$

It follows that

$$AX + XA = (A^2 - B^2) + X^2 \geqslant 0. \tag{8.1}$$

Suppose X is not positive semi-definite, then X has a negative eigenvalue, λ, with eigenvector u. Now, by (8.1),

$$0 \leqslant (u \cdot (AX + XA)\, u) = 2(u \cdot AXu) = 2\lambda(u \cdot Au)$$

which contradicts the facts that λ is negative and A positive definite.

We are now ready to prove Lemma 1. Let $C^{1/2}$ be any Hermitian square root of C. If $ACA \geqslant BCB$ then $C^{1/2}ACAC^{1/2} \geqslant C^{1/2}BCBC^{1/2}$. This implies that $(C^{1/2}AC^{1/2})^2 \geqslant (C^{1/2}BC^{1/2})^2$ and hence that

$$C^{1/2}AC^{1/2} \geqslant C^{1/2}BC^{1/2}. \tag{8.2}$$

Now C is nonsingular and so therefore is $C^{1/2}$. We multiply (8.2) on the left and right by $C^{-1/2}$ and the result follows.

Proof of Lemma 2. If $\int_a^b f(s)^2\, ds = 0$ then $f = 0$ almost everywhere and there is nothing to prove. Suppose $\int_a^b f(s)^2\, ds \neq 0$. Let

$$c := \int_a^b f(s)^2\, ds,$$

$$\bar{A} := \int_a^b f(s)\, A(s)\, ds.$$

We note that $0 \leqslant \int_a^b (cA - f\bar{A})^2\, ds$. An immediate consequence of this is the relation

$$\begin{aligned} 0 \leqslant c^2 \int_a^b A(s)^2\, ds &- c\bar{A} \int_a^b f(s)\, A(s)\, ds \\ &- c\left(\int_a^b f(s)\, A(s)\, ds\right) \bar{A} + \bar{A}^2 \int_a^b f(s)^2\, ds. \end{aligned} \tag{8.3}$$

From (8.3) we see that

$$0 \leqslant \left(c^2 \int_a^b A(s)^2 \, ds \right) - c\bar{A}^2.$$

Since $c \neq 0$ the result follows.

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